APPENDIX: THE MODEL FOR SIMULATIONS[15]

A1: Structure of CDOs: Seniority and Resecuritization

CDOs are special purpose vehicles that hold portfolios of assets and issues securities backed by the cash flows from these assets. Since the early 2000s, CDO markets have been growing rapidly. As subprime loans were securitized actively, the number of CDOs that had a portfolio of these subprime RMBSs significantly increased.

Both ABSs and CDOs have multi-tiered liability structures. I created a hypothetical RMBS by collecting 1,000 mortgages and sliced them into three tranches: senior, mezzanine, and equity. A new CDO was created by constructing a portfolio of 10 RMBS mezzanine tranches. This CDO cash flow should be distributed according to the prespecified seniority of senior, mezzanine, and equity tranches.

A2: Outline of the Model

Let $L_i(t)$ be the amount of loss incurred by loan i at time t; A_i, the amount of the principal of loan i; LGD_i, the loss given the default of loan i; and τ_i, the timing of default of loan i. Then, $L_i(t)$ is written as follows:

$$L_i(t) = A_i \times LGD_i \times \iota_{\{\tau_i \le t\}} \qquad (1)$$

where ι is an index function that takes the value of 1 if t is later than the time of default τ, and 0 otherwise. For simplicity, no prepayments are considered. The total loss L of the loan pool that consists of N loans is

$$L(t) = \sum_{i=1}^{N} L_i(t) = \sum_{i=1}^{N} A_i \times LGD_i \times \iota_{\{\tau_i \le t\}}. \qquad (2)$$

In order to obtain a loss distribution from equation (2), given the information of A_i, it is necessary to make specific assumptions on LGD_i and τ_i. In general, loan defaults are dependent not only on the idiosyncratic risks of each debtor but also on the macroeconomy. Thus, loan defaults are mutually dependent and it is necessary to know the joint distribution of the timing of the defaults of all loans, $G(\tau_1, \cdots, \tau_N)$.

Let $G_i(\tau_i)$ be a marginal distribution of τ_i, and I assume the one-factor Gaussian copula model. That is, if a common factor V is given, then the default timing of each individual loan $\tau_1, \cdots, \tau_N$, where $i(i = 1, \cdots, N)$ is assumed to be independent. Under this assumption, the marginal distribution, $G_i(\tau_i | V)$, is written as follows;

$$G(\tau_1, \cdots, \tau_N | V) = \prod_{i=1}^{N} G_i(\tau_i | V) \qquad (3)$$

With this assumption, once I know $G_i(\tau_i | V)$, then I can obtain the joint distribution of losses of the entire pool. It is noted that the joint distribution of the default timing, $G(\tau_1, \cdots, \tau_N | V)$ can be obtained as an integral of a common factor V with their distribution function G_V. That is,

[15] The simulation shown in this appendix is based on Fujii and Takemoto (2009) in which more details are explained.

$$G(\tau_1,\cdots,\tau_N) = \int_{-\infty}^{\infty} G(\tau_1,\cdots,\tau_N \mid V=v)\,dG_V(v)$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} G_i(\tau_i \mid V=v)\,dG_V(v). \qquad (4)$$

To specify the marginal distribution of each loan, I introduce the latent variable X_i, which follows standard normal distribution, and assume the following for τ_i:

$$\tau_i = G_i^{-1}(\Phi(X_i))$$

where Φ represents a distribution function of standard normal distribution. Given a common factor V , if $X_1,\cdots,X_N$ are conditionally independent, then $\tau_1,\cdots,\tau_N$ are also conditionally independent. I model X_i as shown in equation (5):

$$X_i = \sqrt{\rho_i}V + \sqrt{1-\rho_i}\varepsilon_i \quad (i=1,\cdots,N) \qquad (5)$$

where a common factor V is assumed to follow the standard normal distribution and an idiosyncratic risk factor ε_i, which is independent of V, is assumed to follow the standard normal distribution[16].

Parameter $\rho_i \in [0,1]$ represents a correlation among the latent variables X_is[17]. Since the variable X_i determines the default timing, it could be interpreted as implying the debtor i's asset value and ρ_i is often referred to asset correlation. It should be noted that the correlation between the latent variables （$Corr(X_i,X_j)$） has a one-to-one correspondence with the correlation of the default timing ($Corr(\tau_i,\tau_j)$） ; however, the values are not the same.

For $G_i(\tau_i \mid V)$, I put [18]

$$G_i(\tau_i \mid V=v) = Prob\{X_i \le \Phi^{-1}(G_i(\tau_i)) \mid V=v\}$$

$$= Prob\left\{\varepsilon_i \le \frac{\Phi^{-1}(G_i(\tau_i)) - \sqrt{\rho_i}v}{\sqrt{1-\rho_i}}\right\}$$

$$= \Phi\left(\frac{\Phi^{-1}(G_i(\tau_i)) - \sqrt{\rho_i}v}{\sqrt{1-\rho_i}}\right). \qquad (6)$$

In the model described above, the following three assumptions determine the loss distribution of the pool: i) loss given the default of each loan i : $LGD_i (i=1,\cdots,N)$, ii) marginal distribution of the default timing of each loan: $G_i(\cdot)(i=1,\cdots,N)$, iii) correlation parameter among the default timing of each loan : $\rho_i(i=1,\cdots,N)$. Hereafter ρ_i is referred to

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[16] $G_i(\tau_i)$ follows uniform distribution of $[0,1]$; therefore, $X_i = \Phi^{-1}(G_i(\tau_i))$ follows standard normal distribution.

[17] $Corr(X_i,X_j) = \sqrt{\rho_i\rho_j}$; thus, once I assume a specific value of ρ , I have $Corr(X_i,X_j) = \rho$. Sometimes, by writing $\rho_i = a_i^2$, $X_i = a_i V + \sqrt{1-a_i^2}\varepsilon_i$ is modeled. If this is the case, for example, if ρ_i is equal to 0.1, then a_i is equal to approximately 0.3.

[18] Plugging equation (6) into equation (4); as a result, the distribution of the default timing becomes Gaussian copula（Li. 2000）.

"default correlation" and, for simplicity, it is assumed to be the same for all loans regardless of i.

A3: Loss Calculations of RMBSs and CDOs

Cash flows arising from the underlying assets are distributed according to the prespecified seniority. In our simulation example, $N = 1,000$ is assumed and these loans are assumed to be homogeneous. For simplicity of calculation, a coupon is set to be 0. In tranching, the senior tranche is defined as a tranche that has 1% chance of default. 10% of the principal is set aside as equity tranche, and thus, the remaining part is rated as mezzanine. The Monte Carlo simulation is conducted 10,000 times, and the multiply-with-carry algorithm is used for quasi-random sampling.

The loss distribution is plotted at the timing of maturity, which is 5 years. In the case of the first-stage RMBS, the 99% point of cumulative density coincides with the loss rate of 20%; therefore, approximately 80% of the principal could be rated as AAA through this seniority structure. In the second-stage securitization for creating a CDO, 10 mezzanine tranches of RMBSs are pooled. This time, the 99% point of cumulative density coincides with the loss rate of 70%; therefore, approximately 30% of the principal could be rated as AAA through this seniority structure. As a result, 60% of the CDO principal is rated as mezzanine under the assumption that 10% of the principal is set aside as equity portion.